

Planar Ising Model Observables and Non-Backtracking Walks

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Abstract

This paper presents an alternative proof of the connection between the Ising model on a planar graphs G and the set of non-backtracking walks on G first reported in [She60]. The techniques used give concise formulas for the spin-spin correlation functions of the Ising model without boundary conditions and the Ising model with plus boundary conditions, and as a corollary of these formulas the spinor holomorphic fermion observable used in [CHI12] is naturally identified. The paper is combinatorial and makes use of Viennot's theory of heaps of pieces [Vie86].

1 Introduction

The connection between the Ising model and non-backtracking walks began with a paper by Sherman [She60] that made rigorous the Kac-Ward [KW52] approach to the Ising model on \mathbb{Z}^2 . Roughly contemporaneous to Sherman's work was that of Vdovichenko [Vdo65, Mus10]. Several papers discussing the work of Sherman and Vdovichenko have been published in the intervening years; of particular note is an exposition by Bourgoynne [Bur63]. More recently Loebl [Loe04] and Cima-soni [Cim10] have obtained generalizations of the expressions of Sherman and Kac-Ward that apply to arbitrary finite graphs by embedding them in closed orientable surfaces of sufficiently high genus.

The new aspects of this paper are that expressions for correlation functions are derived in addition to expressions for the free energy, that the paper treats all planar graphs on equal footing, and that the proof utilizes established tools of combinatorics, namely the theory of heaps of pieces introduced by Viennot [Vie86]. Moreover, the expressions derived for the correlations naturally identify a key component of recent works on the Ising model, the spinor holomorphic fermion used by [CHI12]. The rest of the introduction summarizes the results of the paper, leaving proofs for the remaining sections. No prior knowledge of the Ising model is needed to understand the main result of the paper.

A graph whose vertices all have even degree is called *even*. The central result of this paper is a new proof of the fact that the generating function for even subgraphs of a finite planar graph G can be expressed as the exponential of a sum of weighted non-backtracking walks in the graph G . Some terminology will be needed. A *walk* $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ is a sequence of adjacent vertices in a graph, and $|\gamma|$ is the *length* of γ . A walk is *non-backtracking* if $\gamma_j \neq \gamma_{j+2}$ for $1 \leq j \leq |\gamma| - 2$. Let $\Gamma_{\text{nb}}(H, x, y)$ denote the set of non-backtracking walks on a planar graph H that begin at x and end at y , and $\Gamma_{\text{nb}}(H) = \cup_x \Gamma_{\text{nb}}(H, x, x)$. For $x \neq z$ let $\angle(x, y, z)$ denote the angle in $(-\pi, \pi)$ between the vectors $y - x$ and $z - y$ when H is embedded in the plane; see Figure 1.

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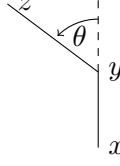


Figure 1: The turning angle $\theta = \angle(x, y, z)$ is the exterior angle of the polygonal segment (x, y, z) .

Define a weight on non-backtracking walks by

$$w(\gamma) \equiv - \prod_{j=1}^{|\gamma|} K_{\gamma_j \gamma_{j+1}} e^{i \frac{\angle(\gamma_j, \gamma_{j+1}, \gamma_{j+2})}{2}}, \quad (1)$$

where $\{K_{xy} \mid xy \in E(G)\}$ is a collection of indeterminates and i is the imaginary unit. In the definition of $w(\gamma)$ the indices of the vertices γ_j are taken mod $|\gamma|$, i.e., $\gamma_{|\gamma|+j} \equiv \gamma_j$.

Theorem. *The generating function of even subgraphs of a finite planar graph G can be expressed as*

$$\sum_{\substack{H \subset G \\ H \text{ even}}} \prod_{xy \in E(H)} K_{xy} = \exp \left(\sum_{\gamma \in \Gamma_{\text{nb}}(G)} \frac{w(\gamma)}{2^{|\gamma|}} \right). \quad (2)$$

The Ising model on a graph $G = (V, E)$ is the probability measure $\langle \cdot \rangle_L$ on configurations $\sigma \in \{\pm 1\}^{|V|}$ given by

$$\mathbb{P}(\sigma) \equiv \frac{1}{Z} \prod_{xy \in E} \exp(L_{xy} \sigma_x \sigma_y) / Z, \quad Z \equiv \sum_{\sigma \in \{\pm 1\}^{|V|}} \prod_{xy \in E} \exp(L_{xy} \sigma_x \sigma_y). \quad (3)$$

The value σ_x for $x \in V(G)$ is called the *spin* at the vertex x , and Z is called the *partition function* for the Ising model on the graph G . The quantities $L_{xy} \in \mathbb{C}$ are called couplings. There is another closely related model, called the Ising model with plus boundary conditions. This is the measure $\langle \cdot \rangle_{L, \text{plus}}$ defined by the same equations, but with the sums restricted to $\sigma_x = 1$ for a specified set $\partial V \subset V$ of *boundary vertices*. The connection between the Ising model on a graph G and even subgraphs of G is established by the so-called high temperature expansion:

Proposition 1.1. *The partition function Z of the Ising model on the graph G can be expressed as*

$$Z = 2^{|V|} \prod_{e \in E} \cosh K_e \sum_{\substack{H \subset G \\ H \text{ even}}} \prod_{xy \in E(H)} \tanh L_{xy}, \quad (4)$$

where H even means $\deg(v)$ is even for all $v \in V(H)$.

Proof. See [Bax82]. □

Hence, for applications to the Ising model the indeterminates K_{xy} will be set to be equal to $\tanh L_{xy}$. An immediate corollary of Theorem 1 and Proposition 1.1 is an expression for the log of the partition function as a sum over weighted non-backtracking walks; this quantity is called the

free energy of the Ising model, and is the quantity that previous combinatorial approaches to the Ising model have discussed. Two further corollaries of Theorem 1 are expressions for the spin-spin correlation function for both the Ising model and the Ising model with plus boundary conditions on planar graphs; for the rest of this section assume G is planar. The precise statements of the corollaries and the definition of the winding number of a non-closed walk are given in Section 5. Let $\mathcal{W}(\gamma, z^*)$ denote the index (or winding number) of a path γ about $z^* \in V(G^*)$, where G^* is the graph dual to G . For $x^*, y^* \in V(G^*)$ a walk $\gamma \in \Gamma_{\text{nb}}$ is called x^*, y^* odd if $\mathcal{W}(\gamma, x^*) + \mathcal{W}(\gamma, y^*)$ is odd.

Corollary 1.2. *Let G be a finite planar graph and $x^*, y^* \in V(G^*)$ denote dual vertices corresponding to an arbitrary choice of faces adjacent to $x, y \in V(G)$. The spin-spin correlation $\langle \sigma_x \sigma_y \rangle_L$ for the Ising model on G with couplings $L \equiv \{L_{xy}\}$ is*

$$\langle \sigma_x \sigma_y \rangle_L = \langle \mu_{x^*} \mu_{y^*} \rangle_L \sum_{\gamma \in \Gamma_{\text{nb}}(G, x, y)} \bar{w}(\gamma) (-1)^{\mathcal{W}(\gamma, x^*) + \mathcal{W}(\gamma, y^*)}, \quad (5)$$

where $K_{xy} = \tanh L_{xy}$, $\bar{w}(\gamma) = c_{ab} w(\gamma)$, where c_{ab} is an explicitly computable unit modulus constant, and

$$\langle \mu_{x^*} \mu_{y^*} \rangle_L \equiv \exp\left(- \sum_{\substack{\gamma \in \Gamma_{\text{nb}}(G) \\ x^*, y^* \text{ odd}}} \frac{\bar{w}(\gamma)}{|\gamma|}\right). \quad (6)$$

Remark 1.3. The quantity $\langle \mu_{x^*} \mu_{y^*} \rangle_L$ is defined by Equation (6), and the symbols μ_x have no independent definition. This notation has been chosen to make a link with the physics literature on the Ising model. No knowledge of this connection is needed to understand this paper.

Let G_{low} denote the graph dual to G , with boundary $\partial V(G_{\text{low}})$ the dual vertex v^* corresponding to the outer face of G .

Corollary 1.4. *Let G be a finite planar graph, and consider the Ising model with plus boundary conditions on G_{low} . If the couplings $L_{x^* y^*}$ are Kramers-Wannier dual to the couplings L_{xy} on G , then the spin-spin correlation for spins at vertices x^*, y^* is given by*

$$\langle \sigma_{x^*} \sigma_{y^*} \rangle_{L^*, \text{plus}} = \exp\left(- \sum_{\substack{\gamma \in \Gamma_{\text{nb}}(G) \\ x^*, y^* \text{ odd}}} \frac{\bar{w}(\gamma)}{|\gamma|}\right), \quad (7)$$

where $K_{xy} = \tanh L_{xy}$ and $\bar{w}(\gamma) = c_{ab} w(\gamma)$, with c_{ab} an explicitly determinable constant of unit modulus.

Combining the corollaries yields

$$\frac{\langle \sigma_x \sigma_y \rangle_L}{\langle \sigma_{x^*} \sigma_{y^*} \rangle_{L^*, \text{plus}}} = \sum_{\gamma \in \Gamma_{\text{nb}}(G, x, y)} \bar{w}(\gamma) (-1)^{\mathcal{W}(\gamma, x^*) + \mathcal{W}(\gamma, y^*)}. \quad (8)$$

Recent work of Chelkak, Hongler, and Izyurov has shown the conformal covariance of multispin correlations in the Ising model at criticality, using what they term spinor holomorphic fermion observables [CI11, CHI12]. Equation (8) identifies the observable in the case of two spins with the

weighted sum of non-backtracking walks on the right-hand side of the equation, which is a different representation of the observable than appears in the literature.

During the preparation of this work the author learned that Corollaries 1.2 and 1.4 on correlation functions have been found independently by Kager, Lis, and Meester [KLM12]. In addition, these authors have obtained convergence results for the correlation expansions as part of a program to provide combinatorial proofs of results about the Ising model that currently rely on the Onsager solution [Ons44].

1.1 Preliminaries and Outline of Paper

A graph will mean a finite graph without loops or multiple edges unless otherwise noted. G will denote a fixed background graph, and $|V(G)|$ and $|E(G)|$ will denote the number of vertices and edges in G . $G^* = (V^*, E^*)$ will denote the graph dual to G when G is planar and embedded. Edges $\{x, y\}$ will be abbreviated xy when it will not cause confusion. It will be assumed that each vertex v of the graph carries a cyclic order on the edges incident to v , and that there is an associated angle $\angle(e_j, e_{j+1}) = -\angle(e_{j+1}, e_j) \in (-\pi, 0]$ such that the sum of the angles at a fixed vertex is 2π , i.e.,

$$\sum_{j=1}^{d_v} \angle(e_j, e_{j+1}) = 2\pi, \quad (9)$$

for all $v \in V(G)$, where $e_{d_v+1} = e_1$ and d_v denotes the degree of the vertex v . The angle $\angle(e_j, e_{j+k})$ is defined to be $\sum_{\ell=j}^{j+k-1} \angle(e_\ell, e_{\ell+1})$.

Remark 1.5. The angles defined above can be thought of as a type of local embedding of the graph into \mathbb{R}^2 at each vertex. This enables turning angles to be assigned to matchings, which are introduced in Section 2.1.

The outline of the rest of the paper is as follows:

- Section 2 introduces loops and shows that the generating function for even subgraphs is the same as a generating function for edge disjoint collections of loops.
- In Section 3 geometric properties of planar curves are used to factorize the weight on collections of loops, and it is shown that the factorized form of the weight is local.
- Section 4 uses the theory of heaps of pieces, along with a combinatorial loop-erasure argument, to show that the generating function of collections of edge disjoint loops is equivalent to the generating function of non-backtracking walks beginning and ending at the same vertex.
- In Section 5 the proof of the main theorem is completed, and corollaries for the Ising model are presented.

Remark 1.6. The Ising model as defined in this paper is occasionally called the *Ising model with free boundary conditions* in the literature. Throughout this paper the Ising model with no qualifiers will refer to the measure defined by Equation (3); for grammatical reasons it will also be called the Ising model without boundary.

2 Decomposing Even Graphs

The main idea in this section is to represent even subgraphs in terms of orientable objects. This is done by choosing, for each vertex v of the subgraph, a matching of the edges incident to v . A choice of matchings decomposes an even subgraph into a collection of closed loops, each loop being formed by picking a sequence of edges that are matched to one another.

Section 2.1 constructs a weight on matchings such that the sum of the weight of matchings corresponding to a given even subgraphs is equal to the weight of the subgraph itself. Section 2.2 shows that matchings are equivalent to loop decompositions of even subgraphs and then transfers the weight on matchings to a weight on collections of loops.

2.1 Matchings of Even Graphs

Definition 2.1. The *line graph* $\mathcal{L}(H)$ of a graph H is the graph with vertices $E(H)$ and edges $\{\{xy, yz\} \mid xy, yz \in E(H)\}$.

The vertices of the line graph will be called *half-edges*. For each vertex $v \in H$ let $\iota(v, H)$ denote the subgraph induced in $\mathcal{L}(H)$ by the set of vertices $\{vw \in E(H)\}$. Figure 2 illustrates an example.

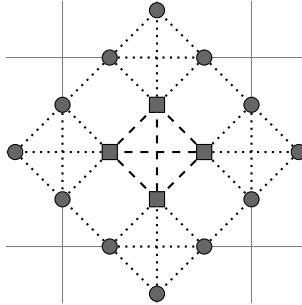


Figure 2: Part of the line graph $\mathcal{L}(H)$ superimposed on H , for $H = \mathbb{Z}^2$. The subgraph $\iota(0, H)$, which is not properly embedded in \mathbb{R}^2 , is comprised of the square vertices and dashed edges.

Definition 2.2. Two edges $e_1 e_2, f_1 f_2 \in \iota(v, H)$ *cross at v* if $e_1 < f_1 < e_2 < f_2$ in the cyclic order on edges at v .

Remark 2.3. The definition of a crossing has a geometric motivation. If H is embedded in the plane, $\iota(v, H)$ is naturally (possibly non-properly) embedded in the plane by placing the vertices of $\iota(v, H)$ at the midpoints of the edges of H . If the cyclic orientation on edges arises from the orientation of the plane, two edges cross if and only if they intersect transversely as line segments in the plane.

Definition 2.4. Let $H = (V, E)$. A *perfect matching* (or *dimer cover*) of H is a collection of edges $m \subset E$ such that for any $v \in V$ there exists a unique $e \in m$ with $v \in e$.

Define the weight $w_c(m)$ of a matching on $\iota(v, H)$ by

$$w_c(m) = (-1)^{\text{cr}(m)} \prod_{xy, yz \in m} \sqrt{K_{xy} K_{yz}}, \quad (10)$$

where $\text{cr}(m)$ denotes the number of edges in the matching that cross at v .

Lemma 2.5. *Let H be an even subgraph of G , and $v \in V(H)$. Then*

$$\sum_{m \in \mathcal{M}(\iota(v, H))} w_c(m) = \prod_{xy \in V(\iota(v, H))} \sqrt{K_{xy}} \quad (11)$$

Proof. Choose an edge incident to the vertex v to get a linear order on vertices adjacent to v from the cyclic order on edges adjacent to v . Let $A_{xv, yv} = \pm \sqrt{K_{xv} K_{yv}}$ for xv, yv in $E(H)$, with $+$ chosen if $x > y$ in the linear order and $-$ otherwise. Set $A_{xv, xv} = 0$. With these definitions the left-hand side of Equation (11) is the Pfaffian of the skew symmetric matrix A . Using the fact that the Pfaffian of $(\xi_j \xi_k B_{jk})$ equals $\prod \xi_j$ times the Pfaffian of the matrix B and that the Pfaffian of the skew-symmetric matrix that is all 1 above the diagonal is 1 proves the claim. See [Ste90] for proofs of these facts as well as the fact that the left-hand side of Equation (11) is the Pfaffian of A . \square

Extend the weight w_c on matchings of subgraphs $\iota(v, H)$ to a weight on sets of matchings multiplicatively, i.e., $w_c(\{m_{v_j}\}_{j=1}^k) \equiv \prod_{j=1}^k w_c(m_{v_j})$. Under this weight, the collection of matchings corresponding to an even subgraph H have the same weight as the subgraph itself. Formally,

Proposition 2.6. *Let H be an even graph. Then*

$$\prod_{xy \in E(H)} K_{xy} = \sum_{(m_v) \in \prod_{v \in V} \mathcal{M}(\iota(v, H))} w_c((m_v)) \quad (12)$$

Proof. Let $V = V(H)$. By distributivity

$$\sum_{(m_v) \in \prod_{v \in V} \mathcal{M}(\iota(v, H))} w_c((m_v)) = \sum_{(m_v) \in \prod_{v \in V} \mathcal{M}(\iota(v, H))} \prod_{v \in V} w_c(m_v) \quad (13)$$

$$= \prod_{v \in V} \sum_{m_v \in \mathcal{M}(\iota(v, H))} w_c(m_v) \quad (14)$$

$$= \prod_{v \in V} \prod_{xy \in V(\iota(v, H))} \sqrt{K_{xy}}, \quad (15)$$

where the final equality follows from Lemma 2.5. Observe that each half edge xy belongs to only the induced subgraphs $\iota(x, H)$ and $\iota(y, H)$, so for each half-edge xy a factor $\sqrt{K_{xy}}$ is contributed exactly twice. \square

2.2 Loops and Perfect Matchings

A *walk* γ in a graph H is a sequence $\gamma = (\gamma_1, \dots, \gamma_k)$ of vertices $\gamma_j \in V(H)$ such that $\gamma_j \gamma_{j+1} \in E(H)$. If $\gamma_1 = a$ and $\gamma_k = b$ then γ is a *walk from a to b in H* . The *length* $|\gamma|$ of a walk is one less than the number of vertices in the walk. A walk is *non-backtracking* if $\gamma_{j-1} \neq \gamma_{j+1}$ for $2 \leq j \leq |\gamma|$ and is *closed* if $\gamma_{|\gamma|+1} = \gamma_1$, and is (edge) *simple* if $\{\gamma_m, \gamma_{m+1}\} = \{\gamma_n, \gamma_{n+1}\}$ implies $m = n$.

Let $\Gamma(H, a, b)$ denote the collection of all walks from a to b in H . Define $\Gamma(H, a) = \Gamma(H, a, a)$, $\Gamma(H) = \cup_a \Gamma(H, a)$, and define $\Gamma_{\text{nb}}(H, a, b)$, $\Gamma_{\text{nb}}(H, a)$ and so on similarly, with the subscript nb indicating the walks are non-backtracking. When the graph H in which the walks take place is clear the H in the notation will be omitted.

Let $\Omega^c(H, a)$ be the set of closed simple walks from a to a in H , and let $\Omega^c(H) \equiv \cup_a \Omega^c(H, a)$. Define two closed simple walks to be equivalent if they are equivalent as cyclic sequences, or if one is the reversal of the other. Denote the set of walks from a to a under this equivalence relation by $\bar{\Omega}^c(a)$. Elements of $\bar{\Omega}^c$ will be called *loops*.

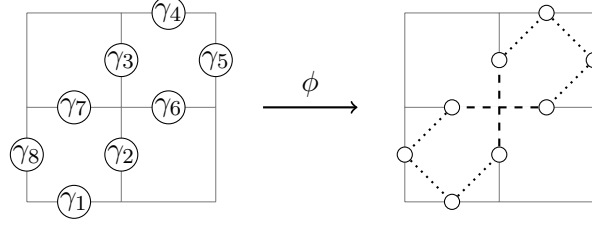


Figure 3: The left-hand side of the figure indicates a walk $\gamma \in \Omega^c$. The right hand side is the collection of perfect matchings $\phi(\gamma)$ associated to γ . In the matchings the dashed edges belong to a copy of K_4 , while the dotted edges are all matchings on separate copies of K_2 .

Definition 2.7. Two subgraphs $H_1, H_2 \subset G$ are *edge disjoint*, written $H_1 \cap_e H_2 = \emptyset$, if $E(H_1) \cap E(H_2) = \emptyset$. Two subgraphs which are not edge disjoint are said to *edge intersect*. $\{H_1, \dots, H_k\}_{\cap_e}$ will denote a set of k pairwise edge disjoint subgraphs.

The notion of edge intersection extends to loops, as loops are just subgraphs with additional structure.

Definition 2.8. Let H be an even graph. A *decomposition* of H is a set $\{C_j\}$ of edge disjoint loops such that $\cup_j E(C_j) = E(H)$. Let $\mathcal{D}(H)$ denote the set of decompositions of an even graph H .

When H is an even graph there is a bijection $\phi: \mathcal{D}(H) \rightarrow \prod_{v \in V(H)} \mathcal{M}(\iota(v, H))$. The bijection is most usefully described in a somewhat informal manner. Let $\gamma \in \Omega^c$ be given by $\gamma = (v_1, \dots, v_k)$. The walk γ defines a matching m_x at each vertex x occurring in γ by pairing the adjacent edges that pass through x , i.e., pairing $v_j v_{j+1}$ to $v_{j+1} v_{j+2}$ if $v_{j+1} = x$; see Figure 3. Let $\phi(\gamma)$ denote the matching produced. Observe that if η is a cyclic shift or reversal of γ then $\phi(\eta) = \phi(\gamma)$, so ϕ can be viewed as a map on the equivalence classes of walks $\bar{\Omega}^c$.

Define ϕ^{-1} as follows. Pick an edge $e = xy$ in a collection of matchings $(m_x) \in \prod_{v \in V(H)} \mathcal{M}(\iota(v, H))$. The matchings at the vertices of e specify two further edges $e_1 = xz_1, e_2 = yz_2$, and the matchings at z_1, z_2 specify two further edges, and so forth. Continuing in this manner produces a loop C_1 . If not all of the edges in the matchings are used, repeat this procedure. It is clear that ϕ^{-1} is the inverse of ϕ .

Extend ϕ to a map on $\mathcal{D}(H)$ by $\phi(\{C_i\}) = \{\phi(C_i)\}$. ϕ^{-1} an inverse for this extended map ϕ as well, as it acts on the image of each loop C_i independently. The preceding discussion will be used in the future, so its conclusion is recorded in a lemma.

Lemma 2.9. *Let $H = (V, E)$ be an even graph. The collection $\mathcal{D}(H)$ of decompositions of H is in bijective correspondence with $\prod_{v \in V} \mathcal{M}(\iota(v, H))$.*

Lemma 2.9 induces a weight on decompositions via $w(\{C_i\}_{\cap_e}) \equiv w_c(\phi(\{C_i\}))$; similarly define $\text{cr}(\{C_i\}_{\cap_e}) \equiv \text{cr}(\phi(\{C_i\}))$. Lemma 2.9 shows that to study the generating function of even subgraphs it suffices to study the generating function of decompositions; henceforth this viewpoint will be adopted. To begin, observe that resolving a collection of matchings into loops distinguishes two types of crossings.

Definition 2.10. Let $C_1, C_2 \in \bar{\Omega}^c$, with $C_1 \cap_e C_2 = \emptyset$. The *self intersection* $C_1 \cdot C_1$ of C_1 is given by

$$C_1 \cdot C_1 \equiv \text{cr}(C_1). \quad (16)$$

The *mutual intersection* $C_1 \cdot C_2$ of edge-disjoint loops C_1 and C_2 is

$$C_1 \cdot C_2 \equiv \text{cr}(\{C_1, C_2\}) - \text{cr}(C_1) - \text{cr}(C_2). \quad (17)$$

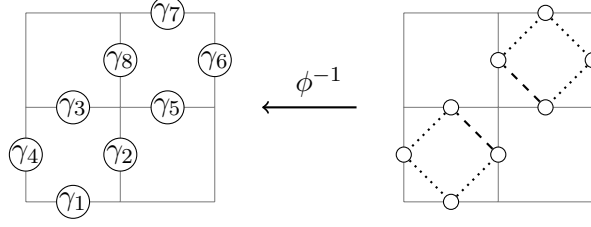


Figure 4: The right-hand side of the figure indicates perfect matchings of a collection of induced subgraphs. The dashed edges belong to a copy of K_4 while the dotted edges belong to distinct copies of K_2 . The left hand side is the element of \mathcal{D} given by ϕ^{-1} .

Lemma 2.11. *Let $\{C_j\}_{\cap_e} \in \mathcal{D}(H)$ be a decomposition of an even graph. Then*

$$\text{cr}(m_{\{C_j\}}) = \sum_{j < k} C_j \cdot C_k + \sum_j C_j \cdot C_j. \quad (18)$$

Proof. As a crossing involves only two edges

$$\begin{aligned} \text{cr}(m_{\{C_j\}}) &= \sum_{j < k} (\text{cr}(m_{C_j} \cup m_{C_k}) - \text{cr}(m_{C_j}) - \text{cr}(m_{C_k})) + \sum_j \text{cr}(m_{C_j}) \\ &= \sum_{j < k} C_j \cdot C_k + \sum_j C_j \cdot C_j. \end{aligned} \quad (19) \quad \square$$

Lemma 2.11 allows us to rewrite the weight on decompositions in a more geometrically intuitive form:

Corollary 2.12. *The weight on decompositions can be written as*

$$w(\{C_j\}_{\cap_e}) = \prod_j (-1)^{C_j \cdot C_j} \prod_{j < k} (-1)^{C_j \cdot C_k} \prod_{xy \in \cup E(C_j)} K_{xy}. \quad (20)$$

Remark 2.13. As mentioned in Remark 2.3, the notion of a crossing has a geometric interpretation when H is an embedded planar graph. In this case, a decomposition $\{C_j\}_{\cap_e}$ joins the vertices of each graph $\iota(v, H)$ along the edges of H , resulting in a set of closed curves in \mathbb{R}^2 . Two edges cross if and only if they intersect as line segments. Slightly perturbing the positions of the vertices of $\iota(v, H)$ ensures that each intersection is a double point. Henceforth when calculating the intersection of walks by viewing them as curves in \mathbb{R}^2 it will be assumed such a perturbation has been performed.

3 Factorization of the Loop Weights

This section presumes that the base graph G is planar, and that the cyclic orientation on edges at each vertex agrees with the orientation given by a straight line embedding of G . In other words, the background graph is taken to be a planar map whose edges are straight lines. Note that such straight line embeddings exist for any planar graph; see [F48]. Further, assume there are two specified vertices $a \neq b \in V(G)$. Utilizing these assumptions allows for significantly more convenient expressions for the weight on decompositions to be derived.

Vertices of G will occasionally be called *primal vertices* to distinguish them from the vertices of the dual graph G^* . A dual vertex x^* is said to be adjacent to a primal vertex x if x^* is associated to a face whose edges contain x . It will be necessary to enlarge the original base graph G into a slightly larger graph G' . Choose, for a and b , adjacent dual vertices a', b' , $a' \neq b'$; these vertices will be called the vertices associated to a, b . The vertices of G' are the vertices of G along with a', b' , and $E(G') = E(G) \cup \{aa', bb'\}$. Let α_{ab} denote a vertex simple walk in G^* from b' to a' ; by an abuse of notation α_{ab} will also denote the reverse of this path when the orientation is contextually indicated.

In this section loops will often be denoted by γ , and walks will be represented as sequences of edges. It will be necessary to extend walks γ on G to walks γ' on G' . If $\gamma \in \Gamma(G, a, b)$, define γ' to be the walk $(a'a, \gamma, bb')$ in G' .

It will also be necessary to specify a point $\bar{x} \in \mathbb{R}^2$ for each point x' such that \bar{x} is located in the face associated to x' and \bar{x} is not located on any edge or vertex in G' . The point \bar{x} will be called the *winding point* associated to x' . See Figure 5.

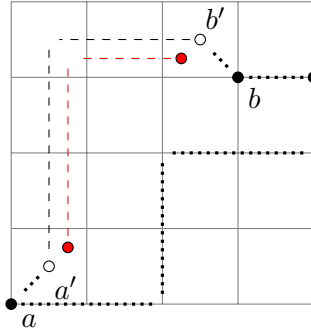


Figure 5: The black dotted path is an extended path γ' from a' to b' . The dashed black path is the chosen dual path α_{ab} between a' and b' that makes extended walks into closed curves. The winding points \bar{a} and \bar{b} used to define winding numbers are shown as red circles, and the dashed red line is a path $\bar{\alpha}_{ab}$ as occurs in the proof of Proposition 3.11.

Note that extended walks define a sequence of matchings of half-edges in $\iota(v, G')$ for $v \in \gamma$, and hence it makes sense, using the terminology of Section 2.1, to speak of the self-intersection of γ' , as well as the intersection of γ' with another walk η when γ' and η are edge disjoint.

A theorem of Whitney on the double points of plane curves will be used to exploit the assumption that G is planar; to state the theorem requires a definition.

Definition 3.1. The *turning number* $\tau(\gamma)$ of a closed smooth curve γ is the number of rotations made by $\gamma'(t)/|\gamma'(t)|$ about the unit circle as t moves from 0 to 1.

Theorem 3.2 (Whitney [Whi37]). *Let γ be a smooth closed curve in \mathbb{R}^2 with at most double points. Then*

$$\tau(\gamma) = \gamma \cdot \gamma - 1 \pmod{2} \quad (21)$$

Definition 3.3. The turning number of a loop is defined to be the turning number of a representative.

That is, if a loop $\bar{\gamma}$ has a representative $\gamma = (\gamma_1, \dots, \gamma_k) \in \Omega^c(G)$ then

$$\tau(\bar{\gamma}) \equiv \left| \frac{1}{2\pi} \sum_{j=1}^k \angle(\gamma_j, \gamma_{j+1}) \right|, \quad \gamma_{k+1} \equiv \gamma_1. \quad (22)$$

To see that the definition of the turning number of a loop is well defined it is enough to note that it only depends on the adjacent edges in a loop, which coincide for all representatives. The sign of result of the sum inside the absolute values on the right-hand side of (22) will depend on the representative γ chosen, but the magnitude will not.

Remark 3.4. If γ is a piecewise smooth curve, then one can smooth γ to a smooth curve and the turning number of the resulting curve is equivalent to the turning number obtained by defining the rotation of $\gamma'(t)/|\gamma(t)|$ to be the angle between the left and right derivatives at points where these two derivatives differ.

Theorem 3.2 also holds for loops. This is because the number of intersections in a loop is equal to the number of intersections of the loop when treated as a curve in \mathbb{R}^2 as mentioned in Remark 2.13, and the turning number of the loop is equal, mod 2, to the turning number of the curves by Remark 3.4. This shows Theorem 3.2 holds as the theorem only depends on the turning number of a curve mod 2.

Proposition 3.5. *Let γ be a loop in a planar graph G . Then*

$$(-1)^{\tau(\gamma)} = \prod_{j=1}^k \exp\left(\frac{i\pi}{2} \angle(\gamma_j, \gamma_{j+1})\right), \quad \gamma_{| \gamma | + 1} \equiv \gamma_1. \quad (23)$$

Let $\mathcal{W}(\eta, x)$ denote the index (or winding number) of a closed curve η about the point x as defined by Equation (46) below. The next proposition will be of use in calculating the mutual and self-intersection of loops.

Proposition 3.6. *Assume $\gamma \in \Omega_{\text{nb}}(H, a, b)$, $a' \neq b'$, $\eta \in \Gamma_{\text{nb}}(H)$, and that γ and η are edge disjoint. Then*

$$(-1)^{\gamma' \cdot \eta} = (-1)^{\mathcal{W}(\eta, a') + \mathcal{W}(\eta, b')}. \quad (24)$$

Proof. The equality follows from a standard technique for computing the winding number of a closed curve γ about x , namely counting the number of intersections between γ and any path ν from x to infinity, provided γ and ν are in general position. Hence, taking ν_1 to be a curve in \mathbb{R}^2 from b' to infinity that is in general position with respect to η that connects with γ' and setting ν_2 to be γ' followed by ν_1 gives

$$(-1)^{\mathcal{W}(\eta, a') + \mathcal{W}(\eta, b')} = (-1)^{\eta \cdot \nu_2 + \eta \cdot \nu_1} \quad (25)$$

$$= (-1)^{\eta \cdot \gamma' + 2\eta \cdot \nu_1} \quad (26)$$

$$= (-1)^{\eta \cdot \gamma'} \quad \square$$

Remark 3.7. In Proposition 3.6 the points a' and b' can be replaced by \bar{a} and \bar{b} by the constancy of $\mathcal{W}(\eta, x)$ on components of $\mathbb{R}^2 \setminus \eta$ as a function of x .

Corollary 3.8. *Let $\gamma_1, \gamma_2 \in \Gamma_{\text{nb}}(H)$ be edge disjoint. Then $\gamma_1 \cdot \gamma_2 = 0 \text{ mod } 2$.*

Proof. The mutual intersection is not changed by viewing the walks as being on a graph with extra vertices. Introduce two extra vertices v_1, v_2 at distinct points on a single edge of γ_1 and consider γ_1 to be a path from v_1 to v_2 . The walk γ_2 intersects this new walk exactly as many times as it does γ_1 . Applying Proposition 3.6 and observing $\mathcal{W}(\gamma_2, v'_1) = \mathcal{W}(\gamma_2, v'_2)$ as edge disjointness implies v'_1 and v'_2 are in the same component of $\mathbb{R}^2 \setminus \gamma_2$. \square

Definition 3.9. The *loop weight* $w(\gamma)$ is given by

$$w(\gamma) = (-1)^{\tau(\gamma)+1} \prod_{j=1}^{|\gamma|} K_{\gamma_j \gamma_{j+1}}. \quad (27)$$

Applying Theorem 3.2 and Corollaries 2.12 and 3.8 to Proposition 2.6 gives a representation for the generating function of even subgraphs that is the basis for the rest of the paper:

Theorem 3.10. *Let G be a planar graph. Then the generating function of even subgraphs is given by*

$$\sum_{\substack{H \subset G \\ H \text{ even}}} \prod_{xy \in E(H)} K_{xy} = \sum_{\{\gamma_j\} \cap e \in \mathcal{D}(H)} \prod_{j=1} w(\gamma_j). \quad (28)$$

For the derivation of correlation formulas it will be necessary to have a formula for the self-intersection of an extended walk from a' to b' . Recall α_{ab} denotes the vertex simple dual walk from b' to a' , and let $\gamma' \alpha_{ab}$ denote the concatenation of γ' with α_{ab} ; this is a closed curve beginning and ending at a' . Another enlarged graph will be needed. By hypothesis G' is a planar map, and hence there is an associated embedding of α_{ab} into G^* . Let G_α be the graph formed by taking the union of G' and α_{ab} , with a vertex at each vertex of G' , at each vertex of α_{ab} , and at each intersection between the edges of α_{ab} and the edges of G' when the edges are considered line segments in \mathbb{R}^2 .

Proposition 3.11. *Let $\gamma \in \Omega_{\text{nb}}(a, b)$, $a' \neq b'$. Then*

$$(-1)^{\gamma' \cdot \gamma'} = (-1)^{\tau(\gamma' \alpha_{ab}) + \mathcal{W}(\gamma' \alpha_{ab}, \bar{a}) + \mathcal{W}(\gamma' \alpha_{ab}, \bar{b}) + 1} \quad (29)$$

where the right-hand side is computed in the graph G_α .

Proof. The self-intersections of $\gamma' \alpha_{ab}$ are either self-intersections of γ' with itself or intersections between α_{ab} and γ' , as α_{ab} is simple. Hence it suffices to compute $\gamma' \alpha_{ab} \cdot \gamma' \alpha_{ab}$ and subtract $\gamma' \cdot \alpha_{ab}$. The first quantity is easily computed using Theorem 3.2:

$$\gamma' \alpha_{ab} \cdot \gamma' \alpha_{ab} = \tau(\gamma' \alpha_{ab}) + 1 \pmod{2}. \quad (30)$$

The intersection between γ' and α_{ab} can be computed by noting that a path $\bar{\alpha}_{ab}$ from \bar{a} to \bar{b} can be chosen such that γ' intersects α_{ab} if and only if $\gamma' \alpha_{ab}$ intersects $\bar{\alpha}_{ab}$, and such that $\bar{\alpha}_{ab}$ and α_{ab} do not intersect. To see how to construct $\bar{\alpha}_{ab}$, note that α_{ab} being vertex simple implies that for δ small enough the δ -neighbourhood of α_{ab} is a topological disk; take $\bar{\alpha}_{ab}$ to be a segment of the boundary of the δ -neighbourhood that begins at \bar{a} near a' and ends at \bar{b} near b' . For δ small enough it is clear that crossing α_{ab} can occur only if $\bar{\alpha}_{ab}$ is also crossed. See Figure 5. Hence, applying Proposition 3.6,

$$\gamma' \cdot \alpha_{ab} = \gamma' \alpha_{ab} \cdot \bar{\alpha}_{ab} - \alpha_{ab} \cdot \bar{\alpha}_{ab} \quad (31)$$

$$= \mathcal{W}(\gamma' \alpha_{ab}, \bar{a}) + \mathcal{W}(\gamma' \alpha_{ab}, \bar{b}). \quad (32)$$

\square

4 Totally Nonbacktracking Walks

The theory of heaps of pieces is used throughout this section, though the ideas are sketched below without reference to the theory. A good introduction can be found in either of [Kra06, Vie86]. The basic definitions, terminology, and theorems as used in this paper are recalled in Appendix A.

This section presents a bijection between a class of closed walks and the set of pyramids of loops. The correspondence can be informally described as follows: a closed walk γ is traced, taking note of each time an edge e is used more than once. If e^* is the last edge used multiple times in the walk, then the segment that begins immediately after the second-to-last visit to e^* and ends with the final visit to e^* defines a closed subwalk C_1 of γ . If C_1 is edge simple, remove it from the path, leaving a shorter path γ' . If C_1 is not simple, the path γ is declared invalid. Repeat the above procedure for the path γ' , removing the identified closed walk C_2 if it is simple, and declaring γ to be invalid otherwise. Continuing in this way eventually reduces a closed walk γ to an edge simple closed walk or results in γ being declared invalid.

Provided γ is not declared invalid, the removed closed subwalks $\{C_j\}$ inductively form a partially ordered set by defining a closed subwalk C_1 to be greater than C_2 if C_1 was removed later than C_2 and shares an edge with C_2 , and taking the transitive closure of the relations generated in this way. The remaining closed simple subwalk of γ can then be declared the maximal element of the poset, as whenever a closed simple subwalk is removed, it shares an edge with the remaining path. The partially ordered set generated by this process is a heap of pieces, and is the output of the bijection when given a valid closed walk γ .

To invert the map, first trace the maximal element. This gives an order on the closed simple subwalks that are immediately beneath the maximal element in the partial order. Moreover, it specifies an orientation on those elements. Inductively inserting these closed simple subwalks into the maximal closed simple subwalk in the given order recovers the original path γ . The rest of this section makes the preceding description precise.

The set of heaps of pieces with piece types A and concurrency relation \mathcal{R} will be denoted $\mathcal{H}(A, \mathcal{R})$, the trivial heaps by $\mathcal{T}(A, \mathcal{R})$, and the pyramids by $\mathcal{P}(A, \mathcal{R})$. Labels for heaps of pieces will be denoted by ℓ . $\mathcal{H}^\bullet(\Omega^c, \cap_e)$ and $\mathcal{P}^\bullet(\Omega^c, \cap_e)$ will indicate heaps and pyramids of loops in which the maximal elements are given labels in Ω^c . Walks will be expressed as sequences of edges. By convention $\gamma_{k+1} \equiv \gamma_1$ if $\gamma = (\gamma_1, \dots, \gamma_k) \in \Omega^c$.

Remark 4.1. Heaps of pieces in which the maximal elements are given labels in a different set than the rest of the elements are not the objects of study in the theory of heaps of pieces. However, in this section they will be a useful structure to encode non-backtracking walks, and in Section 5 they will naturally arise after applying the results of heap theory to genuine heaps of pieces.

Definition 4.2. The *loop index* $L_b: \Gamma \rightarrow \mathbb{N}$ is defined by

$$L_b(\gamma) \equiv \max_k \{k \mid \exists! m > k \ni \gamma(k) = \gamma(m)\}, \quad (33)$$

with $\max \emptyset \equiv -\infty$.

Define a second map L_e by $L_e(\gamma) = m$, where m is the index specified in the definition of $L_b(\gamma)$. Define $L_e(\gamma) = -\infty$ if $L_b(\gamma) = -\infty$. As the notation suggests, L_b and L_e indicate the beginning and end of a closed subwalk.

Let $\gamma \in \Gamma_{\text{nb}}$, $H \in \mathcal{H}^\bullet(\bar{\Omega}^c, \cap_e)$, and define $(\gamma, H) \setminus [a, b] \equiv (\gamma', H')$ when $(\gamma_a, \dots, \gamma_{b-1}) \in \Omega^c$ by

$$\gamma' = (\gamma_1, \dots, \gamma_{a-1}, \gamma_b, \dots, \gamma_{|\gamma|}) \quad (34)$$

$$H' = H + (\gamma_a, \dots, \gamma_{b-1}). \quad (35)$$

In the definition of H' the walk $(\gamma_a, \dots, \gamma_{b-1})$ is considered an element of $\mathcal{H}^\bullet(\Omega^c, \cap_e)$ with a single piece. The addition defined for heaps of loops with oriented maximal elements is the standard addition of heaps of loops, with elements that are maximal in the sum retaining their orientations.

Definition 4.3. The *erasure map* $\epsilon: \Gamma_{\text{nb}} \times \mathcal{H}^\bullet(\Omega^c, \cap_e)$ is defined by

$$\epsilon(\gamma, H) \equiv (\gamma, H) \setminus [L_b(\gamma), L_e(\gamma)] \quad (36)$$

Observe that the erasure map fails to be defined on a pair (γ, H) if:

- the loop index map L_b returns a finite value, but $(\gamma_{L_b(\gamma)}, \dots, \gamma_{L_e(\gamma)-1}) \notin \Omega^c$, so $(\gamma, H) \setminus [L_b, L_e)$ is not defined.
- no edge in γ is repeated, i.e., the loop index map L_b returns ∞ .

The first failure of definition is because the walk that is identified contains a backtracking segment — the edge used twice is used in opposite directions. The second failure is because all that remains is an edge simple closed walk. It will be convenient to extend the definition of ϵ so that $\epsilon(\gamma, H) = (\emptyset, H + \gamma)$ when γ is an edge simple closed walk.

Remark 4.4. It is not hard to see that there exist $\gamma \in \Gamma_{\text{nb}}$ such that $\epsilon(\gamma, H)$ is not defined; see Figure 6.

The fact that the loop erasure map is not always defined naturally singles out the subset of Γ_{nb} on which the erasure map can be repeatedly applied. Let $\epsilon^k(\gamma, H)$ denote the k -fold application of ϵ to (γ, H) , i.e.,

$$\epsilon^k(\gamma) = \underbrace{\epsilon \circ \dots \circ \epsilon}_{k \text{ terms}}(\gamma, H), \quad (37)$$

where by convention $\epsilon^0(\gamma, H) = (\gamma, H)$. For future convenience define $\epsilon^k(\gamma) \equiv \epsilon^k(\gamma, \emptyset)$.

Definition 4.5. A path $\gamma \in \Gamma_{\text{nb}}$ is *totally non-backtracking*, denoted $\gamma \in \Gamma_{\text{tnb}}$, if $\epsilon^k(\gamma, \emptyset)$ is defined for all $k \in \mathbb{N}$ such that $\epsilon^{k-1}(\gamma, \emptyset) \neq (\emptyset, H)$. The *index* $\#(\gamma)$ of $\gamma \in \Gamma_{\text{tnb}}$ is the natural number k such that $\epsilon^k(\gamma, \emptyset) = (\emptyset, H)$.

The loop erasure procedure is now used to define a map from Γ_{tnb} to $\mathcal{P}^\bullet(\bar{\Omega}^c, \cap_e)$. Before the definition of the map can be given, a few more definitions are needed.

Definition 4.6. The *walk order* on $E(H)$ induced by a walk $\gamma \in \Gamma_{\text{nb}}(H)$ is given by $e >_\gamma f$ if $\max\{k \mid \gamma_k = e\} > \max\{k \mid \gamma_k = f\}$.

Recall that a maximal element x in a heap in $\mathcal{H}^\bullet(\bar{\Omega}^c, \cap_e)$ has a label in Ω^c , i.e., $\ell(x) = (\ell(x)_1, \dots, \ell(x)_{|\ell(x)|})$ is a closed walk.

Definition 4.7. Let $\gamma \in \Gamma_{\text{tnb}}$, $H \in \mathcal{H}^\bullet(\bar{\Omega}^c, \cap_e)$. Let x, y be maximal pieces in h . Define $y \prec_\gamma x$ if $\ell(y)_1 >_\gamma \ell(x)_1$.

As the erasure map removes loops that occur at the end of γ first, the relation \prec_γ makes loops removed later greater than loops removed earlier.

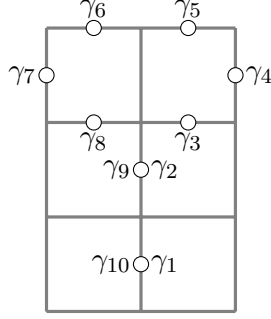


Figure 6: The walk γ illustrated in the figure is a member of Γ_{nb} , but not of Γ_{tnb} . The loop erasure map cannot remove the loop present as it is not simple.

Lemma 4.8. *Let $\gamma \in \Gamma_{\text{tnb}}$. Then $\prec_{\epsilon^k(\gamma)}$ totally orders the maximal pieces of $\epsilon^k(\gamma, \emptyset)$ for $k \leq \#(\gamma)$.*

Proof. No two maximal elements have labels that share an edge by the definition of a heap of loops, and the label of each maximal element shares an edge with the remaining walk. Hence, as the walk order is a total order on the edges that occur in γ , $\prec_{\epsilon^k(\gamma)}$ is a total order on the maximal pieces. \square

Lemma 4.9. *Let $\gamma \in \Gamma_{\text{tnb}}$ and $k \leq \#(\gamma)$. Let ρ_j denote the piece whose label is the loop identified by ϵ^j . Then $\rho_{j_1} \prec_{\epsilon^k(\gamma)} \rho_{j_2}$ only if $j_1 < j_2$.*

Proof. The proof is by induction on k . For $k = 1$ the statement is trivial. Suppose the statement holds for $k = m$. Observe that the removal of the $(m+1)^{\text{st}}$ loop does not change the relative order of ρ_1, \dots, ρ_m , so it suffices to prove $\rho_{m'} \prec_{\epsilon^k(\gamma)} \rho_{m+1}$, where m' is the maximal $j < m+1$ such that ρ_j is a maximal element of the heap in $\epsilon^{m+1}(\gamma)$.

Note that $\ell(\rho_{m+1})_1 \prec_{\epsilon^m(\gamma)} \ell(\rho_m)_1$, as otherwise the loop erasure map would remove $\ell(\rho_{m+1})$ prior to $\ell(\rho_m)$. If $m' = m+1$ the claim follows, as the initial edges of both $\ell(\rho_m)$ and $\ell(\rho_{m+1})$ are retained in $\epsilon^{m+1}(\gamma)$ in the same relative position as in $\epsilon^m(\gamma)$, since these two loops cannot share any edges.

The order $\prec_{\epsilon^m(\gamma)}$ is a total order on $\epsilon^m(\gamma)$, so by transitivity if $m' < m$ then $\ell(\rho_{m+1})_1 \prec_{\epsilon^m(\gamma)} \ell(\rho_{m'})_1$. Noting that $\ell(\rho_{m+1})$ and $\ell(\rho_{m'})$ share no edges, it follows that $\ell(\rho_{m+1})_1$ and $\ell(\rho_{m'})_1$ are both contained in $\epsilon^{m+1}(\gamma)$ in the same relative position as in $\epsilon^m(\gamma)$, so the claim holds. \square

Given $\gamma \in \Gamma_{\text{tnb}}$ Lemma 4.9 defines $\max H$, the maximal piece under $\prec_{\epsilon^k(\gamma)}$ of the maximal pieces of the heap in $\epsilon^k(\gamma, \emptyset)$ for any $k \leq \#(\gamma)$.

Proposition 4.10. *If $\epsilon^{\#(\gamma)}(\gamma) = (\emptyset, H)$, then H is a pyramid.*

Proof. Let $(\gamma', H') = \epsilon^{\#(\gamma)-1}(\gamma)$. Note that the label of each maximal piece in H' contains an edge in γ' , and γ' is an element of Ω^c . It follows that γ' is the unique maximal element in H , so H is a pyramid. \square

Proposition 4.10 defines the map which will be shown to be a bijection, by turning a totally non-backtracking walk into a pyramid with the maximal element a closed walk. Next, the inverse map is constructed. Just as loop erasure was applied multiple times to reduce a walk to a pyramid, the inverse map is defined by repeatedly applying an operation that will be called *loop addition*.

Definition 4.11. Let $\gamma \in \Gamma_{\text{tnb}}$, $H \in \mathcal{H}^\bullet(\bar{\Omega}^c, \cap_e)$, and assume $\max H$ (under \prec_γ) is unique and such that $\omega = \ell(\max H) \cap_e \gamma \neq \emptyset$. Let γ_k be the maximal edge in γ which is contained in ω . Let

$$\gamma + \omega \equiv (\gamma_1, \dots, \gamma_{k-1}, \omega_1, \dots, \omega_{|C|}, \gamma_k, \dots, \gamma_{|\gamma|}), \quad (38)$$

and let $H - \max H$ be the heap of loops formed by removing the maximal piece $\max H$, and orienting the labels of the new maximal elements according to the edge they share with $\gamma + \omega$ that occurs last in $\gamma + \omega$. The *loop addition map* ϵ^{-1} is

$$\epsilon^{-1}(\emptyset, H) \equiv (\max H, H - \max H), \quad (39)$$

$$\epsilon^{-1}(\gamma, H) \equiv (\gamma + \omega, H - \max H), \quad \gamma \neq \emptyset. \quad (40)$$

Lemma 4.12. *Let $\gamma \in \Gamma_{\text{tnb}}$. Then $\epsilon^{-1} \circ \epsilon^{k+1}(\gamma, \emptyset) = \epsilon^k(\gamma, \emptyset)$ for all $k < \#(\gamma)$.*

Proof. Let $\epsilon^{k+1}(\gamma, H) = (\gamma', H')$. First assume γ' is not null. By Lemma 4.9 $\max H$ was removed by the $(k+1)$ st application of ϵ . Hence ϵ^{-1} identifies the correct loop. The orientations of the maximal pieces of H are the orientations induced by the last edge of each maximal piece that is contained in γ , so ϵ^{-1} correctly orients the maximal elements of H by virtue of identifying $\max H$ correctly. \square

Lemma 4.13. *Let $P \in \mathcal{P}^\bullet(\bar{\Omega}^c, \cap_e)$, $k \leq |P|$. Then $\epsilon^{-k}(\emptyset, P) = (\gamma, H)$ is well-defined, $\gamma \in \Gamma_{\text{tnb}}$, and $\epsilon \circ \epsilon^{-k}(\emptyset, P) = \epsilon^{-k+1}(\emptyset, P)$.*

Proof. The claim is immediate for $k = 1$. Suppose $\epsilon^{-k+1}(\emptyset, P) = (\gamma, H)$. By induction the maximal pieces of H have labels that edge intersect γ , so each maximal piece x of H is given a non-zero rank by $\max_e \{e \in \ell(x)\}$. Further, these ranks are distinct, as no two maximal pieces can have labels with an edge in common. Hence the construction used in applying ϵ^{-1} is well defined, as $|H| > 0$ by the hypothesis $k \leq |P|$.

To show that $\gamma \in \Gamma_{\text{tnb}}$ it suffices, by induction, to show $\epsilon \circ \epsilon^{-k} = \epsilon^{-k+1}$. This follows from the construction of ϵ^{-1} and ϵ as $\ell(\max H)$ is the last closed subwalk in $\epsilon^{-k}(\emptyset, P)$. \square

Definition 4.14. Let $\gamma \in \Gamma_{\text{tnb}}$. Define the *loop erasure bijection map* $\tilde{\epsilon}: \Gamma_{\text{tnb}} \rightarrow \mathcal{P}^\bullet(\bar{\Omega}^c, \cap_e)$ by

$$\tilde{\epsilon}(\gamma) = \epsilon^{\#(\gamma)}(\gamma). \quad (41)$$

By Proposition 4.10 the range of $\tilde{\epsilon}$ is the set of pyramids of loops whose maximal elements are closed walk as claimed in the definition. Lemmas 4.12 and 4.13 show that $\tilde{\epsilon}$ is a bijection. Further, it is evident from the constructions given that the set of edges present in a totally non-backtracking walk γ , counting multiplicities, is the same as the set of edges in the labels of the pieces of the pyramid corresponding to γ , again counting multiplicities. Hence the main result of this section has been obtained.

Theorem 4.15. *The map $\tilde{\epsilon}$ is a bijection, and moreover, $|\{k \mid \gamma_k = e\}| = |\{x \in H \mid e \in \ell(x)\}|$ for all $e \in \gamma$.*

Totally non-backtracking walks γ have the property that a cyclic shift of γ or the reversal of γ is still totally non-backtracking. This property is an important ingredient of the proof of Theorem 5.6, and is a corollary of the bijection just presented:

Corollary 4.16. *Let $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|}) \in \Gamma_{\text{tnb}}$. Then any cyclic shift of γ or the reversal of γ is also a totally non-backtracking walk.*

Proof. By Theorem 4.15 γ can be written as a pyramid P with a rooted maximal element. Note that there is a piece $x_e \in P$ corresponding to each edge e in the walk. Picking a walk η that is a cyclic shift of γ or the reversal of γ is equivalent to choosing a particular edge e_η that occurs in γ and an orientation for that edge. Let x_η be the piece of the pyramid such that $e \in \ell(x)$.

As discussed in [Vie86] heaps of pieces are equivalent to acyclic labelled directed graphs, where two vertices are joined by an edge if and only if their labels intersect. If an edge is directed from vertex a to vertex b the piece b lies above a . A pyramid is thus an acyclic labelled directed graph with a unique vertex with outdegree zero.

To prove the corollary, trace η , and each time a new piece is entered, direct this piece towards the previous one. This results in an acyclic directed graph with a unique maximal piece x_η , as each other piece is preceded by another. The graph is acyclic as if adding an edge produced a cycle, the most recently entered piece must already have an incoming edge; but this contradicts the piece being previously unvisited. By Theorem 4.15 the resulting pyramid corresponds to a totally non-backtracking walk. To see that the non-backtracking walk is η follows from the definition of loop addition and the construction of the pyramid. \square

5 Main Theorem and Corollaries for the Ising Model

5.1 Proof of Main Theorem

In this section walks are given as sequences of edges, unless otherwise noted. Define a weight $\tilde{w}: \Gamma_{\text{nb}} \rightarrow \mathbb{C}$ by

$$\tilde{w}(\gamma) \equiv (-1)^{\tau(\gamma)} \prod_{xy \in \gamma} K_{xy} = \prod_{j=1}^{|\gamma|} \exp\left(\frac{i\pi}{2} \angle(\gamma_j, \gamma_{j+1})\right) K_{\gamma_j}. \quad (42)$$

To prove the main result requires lemmas showing the compatibility of the weight on loops with the gluing operation, and hence with the bijection between pyramids of loops and totally non-backtracking walks.

Definition 5.1. Suppose $\gamma^1 \in \Gamma_{\text{nb}}$, $\gamma^2 \in \bar{\Omega}^c$ edge intersect. The *gluing* of γ^1 and γ^2 is the walk γ defined by

$$\gamma = (\gamma_1^1, \dots, \gamma_s^1, \gamma_1^2, \dots, \gamma_{|\gamma^2|}^2, \gamma_{s+1}^1, \dots, \gamma_{|\gamma^1|}^1) \quad (43)$$

where γ_s^1 is the first edge in γ^1 that is contained in the loop γ^2 and $(\gamma_1^2, \dots, \gamma_{|\gamma^2|}^2)$ is the representative of γ^2 with initial edge γ_s^1 .

Remark 5.2. This is just the loop erasure construction of Section 4.

In the next two lemmas it is assumed that the gluing operation is well defined.

Lemma 5.3. *Let γ be the gluing of γ_1 and γ_2 . Then*

$$\tilde{w}(\gamma) = \tilde{w}(\gamma_1) \tilde{w}(\gamma_2). \quad (44)$$

Proof. The weight is a product of factors, one for each pair of adjacent edges in the walk. Gluing preserves the adjacency structure of edges. \square

Lemma 5.4. *Let γ be the gluing of γ_1 and γ_2 . Then*

$$\mathcal{W}(\gamma, \bar{a}) = \mathcal{W}(\gamma_1, \bar{a}) + \mathcal{W}(\gamma_2, \bar{a}) \pmod{2}. \quad (45)$$

Proof. The index of a curve η about \bar{a} is given by

$$\mathcal{W}(\eta, a) = \int_{\eta} \frac{dz}{z - a} \quad (46)$$

with η considered a curve in the complex plane. Splitting the domain of integration γ into its constituent parts γ_1 and γ_2 gives the result. \square

The next lemma is conceptually and computationally helpful as non-backtracking walks are a significantly less constrained set of walks than totally non-backtracking walks are.

Lemma 5.5. *The sum of $\tilde{w}(\gamma)$ over all closed walks that are not totally non-backtracking is zero. That is,*

$$\sum_{\gamma \in \Gamma_{\text{nb}} \setminus \Gamma_{\text{tnb}}} \tilde{w}(\gamma) = 0. \quad (47)$$

Proof. It suffices to exhibit an involution Υ on $\Gamma_{\text{nb}} \setminus \Gamma_{\text{tnb}}$ such that $\tilde{w} \circ \Upsilon = -\tilde{w}$. Suppose the loop erasure map ϵ can be applied k times to γ but not $k + 1$. Define $\Upsilon(\gamma) = \epsilon^{-k} \circ R \circ \epsilon^k(\gamma)$, where R is the map that reverses the first non-simple closed subwalk of a closed walk. That is, if $(\omega_{L_b}, \dots, \omega_{L_e})$ is the first closed subwalk of ω which is not simple then

$$R(\omega) = (\omega_1, \dots, \omega_{L_b-1}, \omega_{L_e}, \omega_{L_e-1}, \dots, \omega_{L_b}, \omega_{L_e+1}, \dots, \omega_{|\omega|}). \quad (48)$$

$R(\omega)$ is an involution and hence Υ is an involution. Let $\omega_{L_b} = xy$. Then $\omega_{L_e} = yx$, and so the turning angle of the reversed non-simple closed subwalk differs from the original by 2π . Hence $\tilde{w}(R(\omega)) = -\tilde{w}(\omega)$. The factorization property of the weight \tilde{w} given by Lemma 5.3 yields $\tilde{w} \circ \Upsilon = -\tilde{w}$. \square

The central theorem of the paper can now be proven. Recall that $w(\gamma) = -\tilde{w}(\gamma)$.

Theorem 5.6. *The logarithm of the generating function of even subgraphs of a finite planar graph G is given by*

$$\log \sum_{T \in \mathcal{T}(\bar{\Omega}^c, \cap_e)} (-1)^{|T|} \tilde{w}(T) = \sum_{\gamma \in \Gamma_{\text{tnb}}} \frac{w(\gamma)}{2|\gamma|}. \quad (49)$$

Proof. That the left-hand side of Equation (49) is the log of the generating function of even subgraphs is the definition of a trivial heap of pieces, combined with Theorem 3.10. Let $Z_G = \sum_{T \in \mathcal{T}(\bar{\Omega}^c(G), \cap_e)} (-1)^{|T|} \tilde{w}(T)$. Note that for $xy \in E(G)$

$$\log Z_G \Big|_{K_{xy}=0} = \log Z_{G \setminus xy}, \quad (50)$$

where $G \setminus xy$ is the graph G with the edge xy removed. The fundamental theorem of calculus then implies

$$\log Z_G = \int_0^{K_{xy}} \frac{d}{d\tilde{K}_{xy}} \log \tilde{Z}_G d\tilde{K}_{xy} + \log Z_{G \setminus xy}, \quad (51)$$

where \tilde{Z}_G is a function of the dummy variables \tilde{K}_{xy} .

The derivative of $\log Z_G$ can be identified with a class of pyramids in \mathcal{P}^\bullet :

$$\frac{d}{dK_{xy}} \log Z_G = -\frac{1}{K_{xy} Z_G} \sum_{\substack{C \\ xy \in E(C)}} \tilde{w}(C) \sum_{T \in \mathcal{T}_C} (-1)^{|T|} \tilde{w}(T). \quad (52)$$

The outer sum is over loops C containing xy , and \mathcal{T}_C is the set of trivial heaps whose elements do not intersect the loop C . Applying Theorem A.1 gives

$$\frac{d}{dK_{xy}} \log Z_G = -\frac{1}{K_{xy}} \sum_{H \in \mathcal{H}_C} \tilde{w}(C) \tilde{w}(H). \quad (53)$$

The maximal pieces in the heaps in \mathcal{H}_C edge intersect C , so the sum in equation (53) can be identified with $1/2$ the sum over pyramids whose maximal element is C , rooted at x and oriented towards y or vice versa. Integrating (53) over K_{xy} gives a prefactor of the N_{xy}^{-1} , where N_{xy} is the number of times xy is contained in the pyramid. Applying Theorem 4.15 gives

$$\int_0^{K_{xy}} \frac{d}{d\tilde{K}_{xy}} \log \tilde{Z}_G d\tilde{K}_{xy} = \sum_{\substack{\gamma \in \Gamma_{\text{tnb}} \\ \gamma_1 \in \{xy, yx\}}} -\frac{\tilde{w}(\gamma)}{2N_{xy}}, \quad (54)$$

where Lemma 5.3 has been used to write the weight of a pyramid as the weight of a walk.

Note that the prefactor $(2N_{xy})^{-1}$ shows that the sum could be considered to be over totally non-backtracking walks up to cyclic shifts and reversals; note that all of the cyclically shifted/reversed totally non-backtracking walks are still totally non-backtracking by Corollary 4.16. As there are $2|\gamma|$ ways to root and orient a walk, this implies

$$\int_0^{K_{xy}} \frac{d}{d\tilde{K}_{xy}} \log \tilde{Z}_G d\tilde{K}_{xy} = \sum_{\substack{\gamma \in \Gamma_{\text{tnb}} \\ \gamma \ni xy}} \frac{w(\gamma)}{2|\gamma|}, \quad (55)$$

where the condition on γ is that either xy or yx is contained in γ , and $-\tilde{w}(\gamma) = w(\gamma)$ has been used. Applying this argument to $\log Z_{G \setminus xy}$ by selecting another edge wz , and repeating, yields

$$\log Z_G = \sum_{\gamma \in \Gamma_{\text{tnb}}} \frac{w(\gamma)}{2|\gamma|}. \quad (56)$$

Applying Lemma 5.5 allows the index of the sum to be expanded to all non-backtracking walks and completes the proof. \square

5.2 Consequences for the Ising Model

Theorem 5.6 combined with Proposition 1.1, immediately yields a representation of the free energy of the Ising model in terms of non-backtracking walks.

Theorem 5.7. *The extrinsic free energy $F_G = \log Z$ of the Ising model on a finite planar graph G is given by*

$$F_G = |V| \log 2 + \sum_{xy \in E} \log \cosh L_{xy} + \frac{1}{2} \sum_{\gamma \in \Gamma_{\text{nb}}^c(G)} \frac{w(\gamma)}{|\gamma|}. \quad (57)$$

Remark 5.8. By approximating \mathbb{Z}^2 by a finite torus and applying the above formula the Onsager solution to the Ising model can be recovered. See [She60] for details.

The starting point to obtain a formula the correlation function for the Ising model is the high-temperature expansion formula for the correlation. In what follows the notation $\langle \cdot \rangle_L$ will denote the Ising measure with couplings $L = \{L_{yz}\}$, and $\langle \cdot \rangle_{L, \text{plus}}$ will denote the Ising measure with plus boundary conditions with couplings L . The graph G under consideration will be implicitly identified by the set of couplings.

Proposition 5.9. *Let G be a planar graph, $a \neq b \in V(G)$. Then*

$$\langle \sigma_a \sigma_b \rangle_L = \left(\sum_{\substack{H \in \mathcal{E}(G_{ab}) \\ ab \in E(H)}} \prod_{xy \in E(H)} L_{xy} \right) \left(\sum_{H \in \mathcal{E}(G)} \prod_{xy \in E(H)} L_{xy} \right)^{-1}, \quad (58)$$

where G_{ab} is the graph G with an edge ab added if one is not already present, in which case $L_{ab} = 1$.

Proof. This follows from applying the high-temperature expansion to the numerator and the denominator of the expression that defines $\langle \sigma_a \sigma_b \rangle_L$. \square

Recall that γ' is γ begun at a' and ending at b' , α_{ab} is the chosen vertex simple dual path from b' to a' , and $\gamma' \alpha_{ab}$ is the closed walk formed by concatenating these two walks.

Definition 5.10. Let $\gamma \in \Gamma_{\text{nb}}(G, a, b)$. The *winding* $\mathcal{W}(\gamma, x)$ of γ about x is defined to be $\mathcal{W}(\gamma' \alpha_{ab}, x)$. Define a closed walk γ to be \bar{a}, \bar{b} odd if $\mathcal{W}(\gamma, \bar{a}) + \mathcal{W}(\gamma, \bar{b})$ is odd.

Theorem 5.11. *The spin-spin correlation function $\langle \sigma_a \sigma_b \rangle_L$ for $a \neq b$ is given by*

$$\langle \sigma_a \sigma_b \rangle_L = \langle \mu_{a'} \mu_{b'} \rangle_L \sum_{\gamma \in \Gamma_{\text{nb}}(G, a, b)} \bar{w}(\gamma) (-1)^{\mathcal{W}(\gamma, \bar{a}) + \mathcal{W}(\gamma, \bar{b})}, \quad (59)$$

where $\bar{w}(\gamma) = c_{ab} w(\gamma')$, c_{ab} is an explicitly computable modulus one constant, and

$$\langle \mu_{x'} \mu_{y'} \rangle_L \equiv \exp \left(- \sum_{\substack{\gamma \in \Gamma_{\text{nb}}(G) \\ \bar{x}, \bar{y} \text{ odd}}} \frac{\bar{w}(\gamma)}{|\gamma|} \right). \quad (60)$$

Proof. For the proof assume that ab is not already an edge in G . The proof is similar, but easier, in the case $ab \in E(G)$. Let G_{ab} be the graph with the edge ab added, and set $L_{ab} = 1$. Define

$$Z_w(G) = \sum_{T \in \mathcal{T}(\bar{\Omega}^c(G), \cap_e)} (-1)^{|T|} w(T). \quad (61)$$

Corollary 2.12, Proposition 3.6, Proposition 3.11, and Remark 3.7 imply that Equation (58) can be rewritten

$$\langle \sigma_a \sigma_b \rangle = \frac{\sum_{C \in \Omega_{\text{nb}}(G, a, b)} \sum_{T \in \mathcal{T}(\bar{\Omega}^c(G_{ab} \setminus C, \cap_e))} w(C' \alpha_{ab}) (-1)^{\mathcal{W}(C, \bar{a}) + \mathcal{W}(C, \bar{b})} \hat{w}(T) \frac{Z_{\hat{w}}(G)}{Z_w(G)}, \quad (62)$$

where $C'\alpha_{ab}$ is the concatenation of the walk C , extended to begin at a' and end at b' , with the fixed walk α_{ab} from b' to a' . $G_{ab} \setminus C$ is the graph G_{ab} with the edges in C removed, and

$$\hat{w}(\gamma) = (-1)^{\mathcal{W}(\gamma, \bar{a}) + \mathcal{W}(\gamma, \bar{b})} w(\gamma). \quad (63)$$

The constant c_{ab} in the theorem statement is defined by $w(C'\alpha_{ab})/w(C')$; this is constant as α_{ab} is fixed, and allows $w(C)$ to be replaced by $\bar{w}(C)$ in Equation (62).

Note that the first multiplicative factor in Equation (62) is of exactly the form handled in the proof of Theorem 5.6, with w replaced by \hat{w} . This yields, using the same analysis as in the proof of Theorem 5.6, along with the fact that the index factors as given by Lemma 5.4, that

$$\langle \sigma_a \sigma_b \rangle_L = \sum_{\gamma \in \Gamma_{\text{nb}}(a, b)} \bar{w}(\gamma) (-1)^{\mathcal{W}(\gamma, \bar{a}) + \mathcal{W}(\gamma, \bar{b})} \frac{Z_{\hat{w}}(G)}{Z_w(G)}. \quad (64)$$

To see that the final ratio of partition functions is equal to $\langle \mu_{a'} \mu_{b'} \rangle_L$ observe that the numerator and denominator are each the generating function of even subgraphs with different weights, and apply the same analysis as before to these generating functions, again using that the index factors as given in Lemma 5.4. As $\hat{w}(\gamma) - w(\gamma) = 2w(\gamma)$ when γ is \bar{a}, \bar{b} odd, and is zero otherwise, the formula follows. \square

Remark 5.12. In [KC71] the quantities $\mu_{x'}$, called *disorder operators* were defined, and the quantity $\langle \mu_{x'} \mu_{y'} \rangle_L$ was defined to be a *disorder-disorder* correlation.

To conclude this section a proof of the relationship between the disorder-disorder and spin-spin correlations is sketched. This relationship was first proven in [KC71] for the case of $G = \mathbb{Z}^2$.

Recall that given a planar graph G the graph G_{low} is the dual graph G^* with boundary the dual vertex associated to the external face. Using the low-temperature expansion shows that the free energy of the Ising model with plus boundary conditions on G_{low} is a generation function for even subgraphs on G . See [Bax82] for the details of the low temperature expansion.

The next definition was first introduced in a landmark paper by Kramers and Wannier [KW41].

Definition 5.13. If $\{L_{xy}\}, \{L_{xy}^*\}$ are couplings on G and G^* respectively, the couplings are called *Kramers-Wannier dual* if $\exp(-2L_{x^*y^*}) = \tanh L_{xy}$.

Theorem 5.14. *Let $a, b \in V(G)$, $a \neq b$, $a' \neq b'$. The disorder-disorder and spin-spin correlations are dual, meaning*

$$\langle \mu_{a'} \mu_{b'} \rangle_L = \langle \sigma_{a'} \sigma_{b'} \rangle_{L^*, \text{plus}}, \quad (65)$$

*when the couplings $L_{x^*y^*} \equiv L_{xy}^*$ are Kramers-Wannier dual.*

Proof. The method of proof for the low temperature expansion implies that

$$\langle \sigma_{a'} \sigma_{b'} \rangle_{L^*, \text{plus}} = \frac{1}{Z} \left(\sum_{\substack{H \in \mathcal{E}(G) \\ \bar{a}, \bar{b} \text{ even}}} \bar{w}(H) - \sum_{\substack{H \in \mathcal{E}(G) \\ \bar{a}, \bar{b} \text{ odd}}} \bar{w}(H) \right). \quad (66)$$

Rewriting the sums over even subgraphs as sums over decompositions and noting that the sign due to being \bar{a}, \bar{b} even or odd is replicated by a factor

$$(-1)^{\mathcal{W}(C, \bar{a}) + \mathcal{W}(C, \bar{b})} \quad (67)$$

for each loop C in a decomposition gives

$$\langle \sigma_{a'} \sigma_{b'} \rangle_{L^*, \text{plus}} = \frac{\sum_{T \in \mathcal{T}(\bar{\Omega}^c, \cap_e)} \hat{w}(T)}{Z}, \quad (68)$$

which is equal to Equation (60) by using Theorem 5.6. \square

5.3 Identification of the Spinor Holomorphic Fermion Observables

This short section notes that Equation (62) naturally identifies one of the holomorphic fermionic observable used in [CI11, CHI12]. As previously described, $\langle \mu_{a'} \mu_{b'} \rangle$ is what [KC71] defined to be the correlation of disorder operators; they showed that

$$\langle \mu_{a'} \mu_{b'} \rangle_L = \langle \sigma_{a'} \sigma_{b'} \rangle_{L^*, \text{plus}}, \quad (69)$$

where the couplings $\{L^*\}$ are on G_{low} , and the couplings $L_{x^*y^*}$ are the couplings obtained by applying the Kramers–Wannier duality transformation to the couplings L_{xy} . Define

$$\langle \sigma_a \mu_{a'} \sigma_b \mu_{b'} \rangle = \sum_{\gamma \in \Gamma_{\text{nb}}(a, b)} \bar{w}(\gamma) (-1)^{W(\gamma, \bar{a}) + W(\gamma, \bar{b})}, \quad (70)$$

so that Theorem 5.11 can be compactly re-expressed as

$$\langle \sigma_a \sigma_b \rangle_L = \langle \sigma_a \mu_{a'} \sigma_b \mu_{b'} \rangle \langle \mu_{a'} \mu_{b'} \rangle_L \quad (71)$$

$$= \langle \sigma_a \mu_{a'} \sigma_b \mu_{b'} \rangle \langle \sigma_{a'} \sigma_{b'} \rangle_{L^*, \text{plus}}, \quad (72)$$

so

$$\langle \sigma_a \mu_{a'} \sigma_b \mu_{b'} \rangle = \frac{\langle \sigma_a \sigma_b \rangle_L}{\langle \sigma_{a'} \sigma_{b'} \rangle_{L^*, \text{plus}}}. \quad (73)$$

At criticality $L^* = L$, combining this with Equation 2.4 of [CHI12] shows that (up to multiplicative constants) $\langle \sigma_a \mu_{a'} \sigma_b \mu_{b'} \rangle$ is exactly the holomorphic fermionic observable considered in [CHI12].

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A Heaps of Pieces

See either of [Kra06, Vie86] for an introduction to the theory of heaps of pieces. The notation used here is the same, though some of the terminology is slightly different.

A *concurrency relation* is a symmetric and reflexive binary relation. Let \mathcal{R} be a concurrency relation on a set \mathcal{B} . The set \mathcal{B} will be called the set of *piece types*. A heap of pieces (H, \preceq, ℓ) is a triple with $\ell: H \rightarrow \mathcal{B}$ and (H, \preceq) a poset such that

1. If $x, y \in H$ and $\ell(x) \mathcal{R} \ell(y)$ then either $x \preceq y$ or $y \preceq x$.
2. The relation \preceq is the transitive closure of the relations from the previous condition.

The map ℓ is called the *labelling* of the pieces. Given a collection of piece types \mathcal{B} and a concurrency relation \mathcal{R} define

1. $\mathcal{H}(\mathcal{B}, \mathcal{R})$ to be the set of all heaps of pieces,
2. $\mathcal{T}(\mathcal{B}, \mathcal{R})$ to be the set of *trivial heaps* of pieces, i.e., heaps of pieces for which $\ell(x)\mathcal{R}\ell(y)$ for any $x, y \in H$.
3. $\mathcal{P}(\mathcal{B}, \mathcal{R})$ to be the set of *pyramids*, i.e., heaps of pieces which contain a unique maximal element.

There is a natural notion of addition (or composition) of two heaps of pieces. If $(H_i, \preceq_i, \ell_i) \in \mathcal{H}(\mathcal{B}, \mathcal{R})$ then define

$$(H_1, \preceq_1, \ell_1) + (H_2, \preceq_2, \ell_2) = (H_3, \preceq_3, \ell_3) \quad (74)$$

by

1. $H_3 = H_1 \cup H_2$
2. The partial order \preceq_3 is the transitive closure of
 - $v_1 \preceq_3 v_2$ if $v_1 \preceq_1 v_2$,
 - $v_1 \preceq_3 v_2$ if $v_1 \preceq_2 v_2$,
 - $v_1 \preceq_3 v_2$ if $v_1 \in P_1, v_2 \in P_2$, and $\ell_1(v_1)\mathcal{R}\ell_2(v_2)$.

Intuitively the heap (H_2, \preceq_2, ℓ_2) is placed on top of the heap (H_1, \preceq_1, ℓ_1) . Note that this addition is *not* commutative.

If $\mathcal{M} \subset \mathcal{B}$ then $\mathcal{H}_{\mathcal{M}}(\mathcal{B}, \mathcal{R})$ is the set of heaps of pieces with piece types \mathcal{B} whose maximal elements have labels in \mathcal{M} .

A weight function $w: \mathcal{B} \rightarrow \mathbb{C}$ naturally extends to heaps of pieces via

$$w(H) = \prod_{x \in H} w(\ell(x)). \quad (75)$$

The main theorem we need from the theory of heaps of pieces is the following.

Theorem A.1.

$$\sum_{H \in \mathcal{H}_{\mathcal{M}}(\mathcal{B}, \mathcal{R})} w(H) = \frac{\sum_{T \in \mathcal{T}(\mathcal{B} \setminus \mathcal{M}, \mathcal{R})} (-1)^{|T|} w(T)}{\sum_{T \in \mathcal{T}(\mathcal{B}, \mathcal{R})} (-1)^{|T|} w(T)} \quad (76)$$

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